Separation results for disjoint closed sets based on normal cones

Xi Yin Zheng

Department of Mathematics, Yunnan University

运筹学基础与发展论坛

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南方科技大学 SOUTHERN UNIVERSITY OF SCIENCE AND TECHNOLOGY

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1 Preliminaries

Recall that a proper lower semicontinuous function φ on a real Banach space X is Frechet differentiable at $\bar{x} \in \mathrm{dom}(\varphi)$ if there exists $x^* \in X^*$ such that

$$\varphi(x) - \varphi(\bar{x}) - \langle x^*, x - \bar{x} \rangle = o(\|x - \bar{x}\|).$$

Frechet subdifferential:

$$\hat{\partial}\varphi(\bar{x}) = \{x^* \in X^* : \ \varphi(x) - \varphi(\bar{x}) - \langle x^*, x - \bar{x} \rangle \geq o(\|x - \bar{x}\|)\}.$$

$x^* \in \hat{\partial}\varphi(\bar{x}) \iff \forall \varepsilon > 0 \ \exists \ \delta > 0 \ \text{ s.t.}$

$$\langle x^*, x - \bar{x} \rangle \leq \varphi(x) - \varphi(\bar{x}) + \varepsilon\|x - \bar{x}\| \quad \forall x \in B(\bar{x}, \delta).$$

$$\varphi(\bar{x}) = \min_{x \in B(\bar{x},\delta)} \varphi(x) \Longrightarrow 0 \in \hat{\partial}\varphi(\bar{x}).$$

Viscosity subdifferential:

$$\partial^V \varphi(\bar{x}) = \{g'(\bar{x}) : \ \varphi - g \,\text{attains its local minimum at}\, \bar{x}\}.$$

If X is a smooth space, then $\partial^V \varphi(\bar{x}) = \hat{\partial}\varphi(\bar{x})$.

Proximal subdifferential: $x^* \in \partial^p \varphi(\bar{x}) \Longleftrightarrow \exists\, \sigma, \delta \in (0,\ +\infty)$ s.t.

$$\langle x^*, x - \bar{x} \rangle \le \varphi(x) - \varphi(\bar{x}) + \sigma \|x - \bar{x}\|^2 \quad \forall x \in B(\bar{x}, \delta).$$

Limit subdifferential: $\bar{\partial}\varphi(\bar{x}) := \mathrm{Limsup}_{x \to \bar{x}} \hat{\partial}\varphi(x)$
$x^* \in \bar{\partial}\varphi(\bar{x}) \Longleftrightarrow \exists x_n \to \bar{x}\ \&\ \exists x_n^* \xrightarrow{w^*} x^*$ s.t. $x_n^* \in \hat{\partial}\varphi(x_n)\ (\forall n \in \mathbb{N})$.

Clarke subdifferential: $\partial\varphi(\bar{x}) := \{x^* \in X^* :\ \langle x^*, h \rangle \le \varphi^\circ(\bar{x}, h)\ \ \forall h \in X\}$,

$$\varphi^\circ(\bar{x}, h) := \lim_{\varepsilon \to 0^+} \limsup_{x \to_f \bar{x}, t \to 0^+} \inf_{v \in B(h, \varepsilon)} \frac{\varphi(x + tv) - \varphi(x)}{t}.$$

Local Lipschitz property of $\varphi \Longrightarrow \varphi^\circ(\bar{x}, h) = \limsup\limits_{x \to \bar{x}, t \to 0^+} \frac{\varphi(x+th) - \varphi(x)}{t}$.

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1. $\partial^p\varphi(\bar{x}) \subset \partial^V\varphi(\bar{x}) \subset \hat{\partial}\varphi(\bar{x}) \subset \bar{\partial}\varphi(\bar{x}) \subset \partial\varphi(\bar{x})$.

2. If φ is smooth around $\bar{x}$, then $\hat{\partial}\varphi(\bar{x}) = \partial\varphi(\bar{x}) = \{\varphi'(\bar{x})\}$

3. If φ is smooth around $\bar{x}$ and $x \mapsto \varphi'(x)$ is locally Lipschiz at $\bar{x}$, then

$$\partial^p\varphi(\bar{x}) = \partial\varphi(\bar{x}) = \{\varphi'(\bar{x})\}.$$

4. If φ is convex, then

$$\partial^p\varphi(\bar{x}) = \partial\varphi(\bar{x}) = \{x^* \in X^* : \langle x^*, x - \bar{x}\rangle \le \varphi(x) - \varphi(\bar{x}) \ \ \forall x \in X\}.$$

5. If $\dim(X) < \infty$ and φ is locally Lipschitz at $\bar{x} \in \mathrm{dom}(\varphi)$, then

$$\partial\varphi(\bar{x}) = \overline{\mathrm{co}}\left\{\lim_{n\to\infty}\varphi'(x_n) : \ x_n \to x, \ \varphi \text{ is Frechet differentiable at each } x_n\right\}.$$

Theorem I. *Let X be a Banach space and $\varphi, \psi : X \to \mathbb{R} \cup \{+\infty\}$ be proper lower semicontinuous functions. The following statements hold:*
(i) $\mathrm{dom}(\partial\varphi)$ *is dense in* $\mathrm{dom}(\varphi)$.
(ii) If ψ is locally Lipschitzat $\bar{x} \in \mathrm{dom}(\varphi)$, then

$$\partial(\varphi + \psi)(\bar{x}) \subset \partial\varphi(\bar{x}) + \partial\psi(\bar{x}).$$

If $\varphi(x) = -\|x\|$ for all $x \in \ell^1$, then $\mathrm{dom}(\hat{\partial}\varphi) = \mathrm{dom}(\bar{\partial}\varphi) = \emptyset$.

Theorem II. *Let X be an Asplund space and let $\varphi, \psi : X \to \mathbb{R} \cup \{+\infty\}$ be proper lower semicontinuous functions. The following statements hold:*
(i) $\mathrm{dom}(\hat{\partial}\varphi)$ *is dense in* $\mathrm{dom}(\varphi)$.
(ii) If ψ is locally Lipschitz at $\bar{x} \in \mathrm{dom}(\varphi)$, then for any $x^* \in \hat{\partial}(\varphi + \psi)(\bar{x})$ and any $\varepsilon > 0$ there exist $x_1, x_2 \in B(\bar{x}, \varepsilon)$ such that

$$x^* \in \hat{\partial}\varphi(x_1) + \hat{\partial}\psi(x_2) + \varepsilon B_{X^*} \text{ and } |\varphi(x_1) - \varphi(\bar{x})| < \varepsilon$$

and so $\bar{\partial}(\varphi + \psi)(\bar{x}) \subset \bar{\partial}\varphi(\bar{x}) + \bar{\partial}\psi(\bar{x})$.
(iii) $\partial\varphi(\bar{x}) = \mathrm{cl}^{w^*}\left(\mathrm{co}\left(\bar{\partial}\varphi(\bar{x}) + \bar{\partial}^{\infty}\varphi(\bar{x})\right)\right)$.

A—a closed set in a Banach space X, $a \in A$.

Bouligand tangent cone:

$$T(A,a) = \{h \in X :\ \exists t_n \to 0^+ \,\&\, \exists h_n \to h \,\text{s.t.}\, a + t_n h_n \in A \ \ \forall n \in \mathbb{N}\}.$$

Clarke tangent cone:

$$\begin{aligned} T_C(A,a) := \{h \in X :\quad & \forall a_n \overset{A}{\to} a \,\&\, \forall s_n \to 0^+ \ \exists h_n \to h \\ & \text{s.t. } \ a_n + s_n h_n \in A \quad \forall n \in \mathbb{N}\}. \end{aligned}$$

$$T_C(A,a) \subset T(A,a)$$

Clarke normal cone:

$$N_C(A,a) := T_C(A,a)^{\circ} = \{x^* \in X^* :\ \langle x^*, h\rangle \le 0 \ \ \forall h \in T_C(A,a)\}.$$

Frechet normal cone:

$$\hat{N}(A,a) := \left\{ x^* \in X^* : \limsup_{x \xrightarrow{A} a} \frac{\langle x^*, x-a\rangle}{\|x-a\|} \leq 0 \right\}$$

If X is an Asplund space, $\left\{a \in A : \hat{N}(A,a) \neq \{0\}\right\}$ is dense in $\mathrm{bd}(A)$.

Proximal normal cone:

$$\hat{N}^p(A,a) := \left\{ x^* \in X^* : \limsup_{x \xrightarrow{A} a} \frac{\langle x^*, x-a\rangle}{\|x-a\|^2} < +\infty \right\}$$

If X is a Hilbert space, $x^* \in N^P(A,a) \Leftrightarrow a \in P_A(a+tx^*)$ for some $t > 0$, and $\left\{a \in A : N^p(A,a) \neq \{0\}\right\}$ is dense in $\mathrm{bd}(A)$.

Proximal point: a point a is called a proximal point of A if $a \in P_A(x)$ for some $x \in X \setminus A$.

In 2010, Borwein [1] asked the following "most striking" open question: Is it possible that in every reflexive Banach space, the proximal points on $\mathrm{bd}(\Omega)$ are dense in $\mathrm{bd}(\Omega)$?

Limit normal cone: $\bar{N}(A,a) := \operatorname{Limsup}_{x \xrightarrow{A} a} \hat{N}(A,x)$

$$x^* \in \bar{N}(A,a) \Longleftrightarrow \exists x_n \xrightarrow{A} a \,\&\, \exists x_n^* \xrightarrow{w^*} x^* \text{ s.t. } x_n^* \in \hat{N}(A,x_n)\ (\forall n \in \mathbb{N}).$$

$$\hat{N}(A,a) \subset \bar{N}(A,a) \subset N_C(A,a).$$

If X is an Asplund space, then $N_C(A,a) = \mathrm{cl}^{w^*}\left(\mathrm{co}\left(\bar{N}(A,a)\right)\right)$.

$$A \cap B(a,r) = B \cap B(a,r) \Longrightarrow \hat{N}(A,a) = \hat{N}(B,a) \ \& \ N_C(A,a) = N_C(B,a).$$

$$\hat{N}(A,a) = \hat{\partial}\delta_A(a),\ \bar{N}(A,a) = \bar{\partial}\delta_A(a),\ N_C(A,a) = \partial\delta_A(a).$$

If A is convex, then

$$T(A,a) = T_C(A,a) = \mathrm{cl}(\mathbb{R}_+(A-a))$$

and

$$\hat{N}(A,a) = N_C(A,a) = \{x^* \in X^* :\ \langle x^*, a\rangle = \sup_{x\in A}\langle x^*, x\rangle\}.$$

$\varphi : X \to \mathbb{R}\cup\{+\infty\}$—a proper lower semicontinuous function

$$\begin{aligned}\hat{\partial}\varphi(x) &= \{x^* \in X^* :\ (x^*, -1) \in \hat{N}(\mathrm{epi}(\varphi), (x, \varphi(x)))\}\\ \bar{\partial}\varphi(x) &= \{x^* \in X^* :\ (x^*, -1) \in \bar{N}(\mathrm{epi}(\varphi), (x, \varphi(x)))\}\\ \partial\varphi(x) &= \{x^* \in X^* :\ (x^*, -1) \in N_C(\mathrm{epi}(\varphi), (x, \varphi(x)))\},\end{aligned}$$

where $\mathrm{epi}(\varphi) = \{(x,t) \in X \times \mathbb{R} :\ \varphi(x) \leq t\}$.

2 Fuzzy separation theorems for disjoint closed sets

Extremal point: *A common point* $\bar{x}$ *of closed sets* $A_1, \cdots, A_m$ *in a normed space is called an extremal point of these closed sets if there exist a neighborhood* V *of* $\bar{x}$ *and* m *sequences* $x_{1k} \to 0, \cdots, x_{mk} \to 0$ *such that*

$$\bigcap_{i=1}^{m}(A_i - x_{ik}) \cap V = \emptyset \quad \forall k \in \mathbb{N}.$$

Extremal Principle: *Let* $\bar{x}$ *be an extremal point of closed sets* $A_1, \cdots, A_m$ *in an Asplund space* X*. Then for any* $\varepsilon > 0$ *there exist* $a_i \in A_i \cap B(\bar{x}, \varepsilon)$ *such that*

$$x_i^* \in \hat{N}(A_i, a_i) + \varepsilon B_{X^*}, \; i = 1, \cdots, m, \; \sum_{i=1}^{m} x_i^* = 0 \;\text{ and }\; \sum_{i=1}^{m} \|x_i^*\| = 1.$$

Corollary. *Let $\bar{x}$ be an extremal point of closed sets $A_1, \cdots, A_m$ in an Asplund space X, and suppose that all but one of $A_1, \cdots, A_m$ are sequentially normally compact at $\bar{x}$. Then there exist $x_i^* \in \bar{N}(A_i, \bar{x})$, $i = 1, \cdots, m$, such that*

$$x_1^* + \cdots + x_m^* = 0 \ \text{ and } \ \|x_1^*\| + \cdots + \|x_m^*\| = 1.$$

Corollary. *Let $\bar{x}$ be an extremal point of closed sets A_1 and A_2 in an Asplund space X, and suppose that A_1 is sequentially normally compact at $\bar{x}$. Then there exist $x^* \in X^*$ such that*

$$\|x^*\| = 1 \ \text{ and } \ x^* \in \bar{N}(A_1, \bar{x}) \cap -\bar{N}(A_2, \bar{x}).$$

If A_1 and A_2 are convex,

$$x^* \in \bar{N}(A_1, \bar{x}) \cap -\bar{N}(A_2, \bar{x}) \Longleftrightarrow \langle x^*, \bar{x} \rangle = \sup_{x \in A_1} \langle x^*, x \rangle = \inf_{x \in A_2} \langle x^*, x \rangle.$$

Non-intersection index: For closed sets $A_1, \cdots, A_m$, let

$$\gamma(A_1, \cdots, A_m) := \inf \left\{ \sum_{i=1}^{m-1} \|x_i - x_m\| : \; x_i \in A_i, \; i = 1, \cdots, m \right\}.$$

$$\gamma(A_1, A_2) = d(A_1, A_2).$$

$$\bigcap_{i=1}^{m} A_i \neq \emptyset \Longrightarrow \gamma(A_1, \cdots, A_m) = 0.$$

$$\gamma(A_1, \cdots, A_m) > 0 \Longrightarrow \bigcap_{i=1}^{m} A_i = \emptyset.$$

Theorem 2.1 ([Zheng-Ng, SIOPT, 2011]). *Let $A_1, \cdots, A_m$ be closed sets in a Banach space X such that $\bigcap_{i=1}^{m} A_i = \emptyset$. Let $\varepsilon > 0$ and $a_i \in A_i$ $(1 \le i \le m)$ be such that*

$$\sum_{i=1}^{m-1} \|a_i - a_m\| < \gamma(A_1, \cdots, A_m) + \varepsilon.$$

Then, for any $\lambda > 0$, there exist $\tilde{a}_i \in A_i$ and $a_i^* \in N_c(A_i, \tilde{a}_i) + \frac{\varepsilon B_{X^*}}{\lambda}$ such that the following properties hold:

(i) $\sum_{i=1}^{m} \|\tilde{a}_i - a_i\| < \lambda$.

(ii) $\max_{1 \le i \le m-1} \|a_i^*\| = 1$ *and* $\sum_{i=1}^{m} a_i^* = 0$.

(iii) $\sum_{i=1}^{m-1} \langle a_i^*, \tilde{a}_m - \tilde{a}_i \rangle = \sum_{i=1}^{m-1} \|\tilde{a}_i - \tilde{a}_m\|$.

Theorem 2.2. *Let X be an Asplund space and $A_1, \cdots, A_m$ be closed nonempty subsets of X such that $\bigcap_{i=1}^{m} A_i = \emptyset$. Let $\varepsilon > 0$ and $a_i \in A_i$ $(1 \le i \le m)$ be such that*

$$\sum_{i=1}^{m-1} \|a_i - a_n\| < \gamma(A_1, \cdots, A_m) + \varepsilon. \tag{2.1}$$

Then, for any $\lambda > 0$ and any $\rho \in (0, 1)$ there exist $\tilde{a}_i \in A_i$ and $a_i^* \in \hat{N}(A_i, \tilde{a}_i) + \frac{\varepsilon B_{X^*}}{\lambda}$ $(i = 1, \cdots, m)$ such that the following properties hold:

(i) $\sum_{i=1}^{m} \|\tilde{a}_i - a_i\| < \lambda$.

(ii) $\max_{1 \le i \le m-1} \|a_i^*\| = 1$ *and* $\sum_{i=1}^{m} a_i^* = 0$.

(iii) $\rho \sum_{i=1}^{m-1} \|\tilde{a}_i - \tilde{a}_m\| \le \sum_{i=1}^{m-1} \langle a_i^*, \tilde{a}_m - \tilde{a}_i \rangle$.

(i) and (ii) of Theorem 2.2$\Longrightarrow$Extremal Principle.

Corollary 2.1. *Let A and B be closed nonempty sets in a Banach space X such that $A \cap B = \emptyset$. Then, for any $\varepsilon > 0$ there exist $a \in A$, $b \in B$ and $a^* \in X^*$ with $\|a^*\| = 1$ such that*

$$a^* \in N_c(A, a) \cap \big(-N_c(B, b) + \varepsilon B_{X^*}\big)$$

and

$$\|a - b\| = \langle a^*, b - a\rangle < d(A, B) + \varepsilon.$$

Corollary 2.2. *Let A be a closed nonempty set in a Banach (resp. Asplund) space X. Then, for any $x \in X \setminus A$ and any $\varepsilon > 0$, there exist $a \in A$ and $a^* \in N_c(A, a)$ (resp. $a^* \in \hat{N}(A, a)$) such that*

$$\|a^*\| = 1 \text{ and } (1 - \varepsilon)\|x - a\| \le \min\{\langle a^*, x - a\rangle, d(x, A)\}.$$

Corollary 2.3. *Let A and B be closed sets in a Banach (resp. Asplund) space X such that $A \cap B = \emptyset$. Suppose that B is bounded and convex. Then, for any $\varepsilon > 0$, there exist $a \in A$ and $a^* \in N_c(A, a)$ (resp. $a^* \in \hat{N}(A, a)$) such that*

$$\|a^*\| = 1 \ \text{ and } \ d(A, B) - \varepsilon < \inf_{x \in B} \langle a^*, x \rangle - \langle a^*, a \rangle.$$

If, in addition, A is convex, then

$$d(A, B) - \varepsilon < \inf_{x \in B} \langle a^*, x \rangle - \max_{x \in A} \langle a^*, x \rangle.$$

Proof of Theorem 2.2. Define $\varphi : X^m \to \mathbb{R} \cup \{+\infty\}$ as follows

$$\varphi(x_1, \cdots, x_m) := \sum_{i=1}^{m-1} \|x_i - x_m\| + \delta_{A_1 \times \cdots \times A_m}(x_1, \cdots, x_m) \quad \forall (x_1, \cdots, x_m) \in X^m.$$

Then φ is a proper lower semicontinuous function on X^m equipped with the ℓ_1-norm

$$\|(x_1, \cdots, x_m)\| := \sum_{i=1}^{m} \|x_i\| \quad \forall (x_1, \cdots, x_m) \in X^m$$

and (2.1) can be rewritten as

$$\varphi(a_1, \cdots, a_m) < \inf\{\varphi(x_1, \cdots, x_m) : \ (x_1, \cdots, x_m) \in X^m\} + \varepsilon.$$

Take $\varepsilon' \in (0, \ \varepsilon)$ such that

$$\varphi(a_1, \cdots, a_m) < \inf\{\varphi(x_1, \cdots, x_m) : \ (x_1, \cdots, x_m) \in X^m\} + \varepsilon'.$$

Then there exists $\lambda' \in (0, \ \lambda)$ such that $\frac{\varepsilon'}{\lambda'} < \frac{\varepsilon}{\lambda}$. By the Ekeland variational principle, there exists $(\bar{a}_1, \cdots, \bar{a}_m) \in X^m$ such that

$$\|(\bar{a}_1,\cdots,\bar{a}_m)-(a_1,\cdots,a_m)\|<\lambda' \tag{2.2}$$

and

$$\varphi(\bar{a}_1,\cdots,\bar{a}_m)\le\varphi(x_1,\cdots,x_m)+\frac{\varepsilon'}{\lambda'}\sum_{i=1}^{m}\|x_i-\bar{a}_i\| \quad \forall(x_1,\cdots,x_m)\in X^m.$$

Hence $(\bar{a}_1,\cdots,\bar{a}_m)\in A_1\times\cdots\times A_m$ is a minimizer of $\varphi+\frac{\varepsilon'}{\lambda'}\|\cdot-(\bar{a}_1,\cdots,\bar{a}_m)\|_{X^m}$. It follows that $\sigma:=\sum_{i=1}^{m-1}\|\bar{a}_i-\bar{a}_m\|>0$ and

$$\begin{aligned}0 &\in \hat{\partial}\left(\varphi+\frac{\varepsilon'}{\lambda'}\|\cdot-(\bar{a}_1,\cdots,\bar{a}_m)\|_{X^m}\right)(\bar{a}_1,\cdots,\bar{a}_m)\\ &= \hat{\partial}(f+\delta_{A_1\times\cdots\times A_m})(\bar{a}_1,\cdots,\bar{a}_m)\end{aligned} \tag{2.3}$$

where

$$f(x_1,\cdots,x_m):=\sum_{i=1}^{m-1}\|x_i-x_m\|+\frac{\varepsilon'}{\lambda'}\sum_{i=1}^{m}\|x_i-\bar{a}_i\| \quad \forall(x_1,\cdots,x_m)\in X^m.$$

Thus, by (2.3) and Theorem II, for any $\beta \in \left(0,\ \ \min\{\frac{\varepsilon}{\lambda} - \frac{\varepsilon'}{\lambda'}, \lambda - \lambda', \frac{\sigma}{m}\}\right)$ there exist

$$(\bar{x}_1, \cdots, \bar{x}_m), (\tilde{a}_1, \cdots, \tilde{a}_m) \in B_{X^m}((\bar{a}_1, \cdots, \bar{a}_m), \beta) \tag{2.4}$$

such that

$$\begin{aligned} 0 &\in \hat{\partial} f(\bar{x}_1, \cdots, \bar{x}_m) + \hat{\partial}\delta_{A_1 \times \cdots \times A_m}(\tilde{a}_1, \cdots, \tilde{a}_m) + \beta B_{X^*}^m \\ &= \hat{\partial} f(\bar{x}_1, \cdots, \bar{x}_m) + \hat{N}(A_1 \times \cdots \times A_m, (\tilde{a}_1, \cdots, \tilde{a}_m)) + \beta B_{X^*}^m \\ &= \hat{\partial} f(\bar{x}_1, \cdots, \bar{x}_m) + \hat{N}(A_1, \tilde{a}_1) \times \cdots \times \hat{N}(A_m, \tilde{a}_m) + \beta B_{X^*}^m. \end{aligned} \tag{2.5}$$

Exact Separation

Theorem 2.3. *Let $A_1, \cdots, A_m$ be closed sets in a Banach space X such that $\bigcap_{i=1}^{m} A_i = \emptyset$, and suppose that there exist $a_i \in A_i$ ($i = 1, \cdots, m$) such that*

$$\sum_{i=1}^{m-1} \|a_i - a_m\| = \gamma(A_1, \cdots, A_m). \tag{2.6}$$

Then there exist $a_i^* \in X^*$ ($1 \le i \le m$) with the following properties:

(i) $\max_{1 \le i \le m-1} \|a_i^*\| = 1$, $\sum_{i=1}^{m} a_i^* = 0$ *and* $a_i^* \in N_c(A_i, a_i)$ ($i = 1, \cdots, m$).

(ii) $\sum_{i=1}^{m-1} \langle a_i^*, a_m - a_i \rangle = \sum_{i=1}^{m-1} \|a_m - a_i\|$.

Theorem 2.4. *Let $A_1, \cdots, A_m$ be closed sets in an Asplund space X such that $\bigcap_{i=1}^{m} A_i = \emptyset$. Further suppose that A_m is compact. Let $\varepsilon > 0$ and $a_i \in A_i$ $(1 \le i \le m)$ be such that*

$$\sum_{i=1}^{m-1} \|a_i - a_m\| < \gamma(A_1, \cdots, A_m) + \varepsilon.$$

Then, for any $\lambda > 0$ and any $\rho \in (0,\ 1)$ there exist $\tilde{a}_i \in A_i$ and $a_i^* \in X^*$ with the following properties:

(i) $\sum_{i=1}^{m} \|\tilde{a}_i - a_i\| < \lambda$.

(ii) $\max_{1 \le i \le m-1} \|a_i^*\| = 1$, $\sum_{i=1}^{n} a_i^* = 0$ *and* $a_i^* \in \hat{N}(A_i, \tilde{a}_i)$ $(i = 1, \cdots, m)$.

(iii) $\rho \sum_{i=1}^{m-1} \|\tilde{a}_i - \tilde{a}_m\| \le \sum_{i=1}^{m-1} \langle a_i^*, \tilde{a}_m - \tilde{a}_i \rangle$.

3 Convex case

Theorem S1. *Let A and B be convex sets in a normed space X such that $\mathrm{int}(B) \neq \emptyset$ and $A \cap \mathrm{int}(B) = \emptyset$. Then there exists $x^* \in X^* \setminus \{0\}$ such that*

$$\inf_{x \in A} \langle x^*, x \rangle \geq \sup_{x \in B} \langle x^*, x \rangle. \tag{3.7}$$

Theorem S2. *Let A be a compact convex set in a normed space X and let B be a closed convex set in X such that $A \cap B = \emptyset$. Then there exists $x^* \in X^*$ such that*

$$\inf_{x \in A} \langle x^*, x \rangle > \sup_{x \in B} \langle x^*, x \rangle. \tag{3.8}$$

Strict separation property: *a closed convex set A in a normed space X is said to have strict separation property if for every closed convex set B in X with $A \cap B = \emptyset$ there exists $x^* \in X^*$ such that (3.8) holds.*

A compact convex set has trivially the strict separation property.

Theorem GW ([Gau-Wong, PAMS, 1996]). *Let A be a bounded closed convex subset of a normed space such that $\mathrm{int}(A) \neq \emptyset$. Then A has the strict separation property if and only if A is weakly compact.*

Theorem GK ([Gale-Klee, Math. Scan., 1959]). *Let A be a closed convex set in $\mathbb{R}^n$. Then A has the strict separation property if and only if A is continuous, that is,*

$$\sigma_A(x^*) := \sup_{x \in A} \langle x^*, x \rangle = \lim_{u^* \to x^*} \sigma_A(u^*) \quad \forall x^* \in \mathbb{R}^n \setminus \{0\}.$$

Theorem ETZ ([Ernst-Théra-Zalinnescu, JFA, 2005]). *Let A be a closed convex set in a reflexive Banach space. Then A has the strict separation property if and only if A is slice-continuous (i.e., for every closed subspace Y of X, $A \cap Y$ is a continuous set in Y).*

From the view point of optimization, it should be interesting to consider whether or not the linear functional x^* in either (3.7) or (3.8) can attain its infimum and supremum over A and B, respectively. However, even in Euclidean space $\mathbb{R}^2$, there exist two disjoint closed convex sets A and B with $\mathrm{int}(B) \neq \emptyset$ such that they cannot be separated attainably, namely there exists no $y^* \in (\mathbb{R}^2)^* \setminus \{0\}$ satisfying

$$\langle y^*, a\rangle = \inf_{x\in A}\langle y^*, x\rangle \geq \sup_{x\in B}\langle y^*, x\rangle = \langle y^*, b\rangle \text{ for some } (a, b) \in A \times B.$$

Two kinds of attainable separation properties

Definition 3.1. *A closed convex set A in a normed space X is said to have attainable separation property if for every closed convex subset B of X with $\mathrm{int}(B) \neq \emptyset$ and $A \cap \mathrm{int}(B) = \emptyset$ there exist $x^* \in X^* \setminus \{0\}$, $a \in A$ and $b \in B$ such that*

$$\langle x^*, a\rangle = \inf_{x\in A}\langle x^*, x\rangle \geq \sup_{x\in B}\langle x^*, x\rangle = \langle x^*, b\rangle. \tag{3.9}$$

Definition 3.2. *A closed convex set A in a normed space X is said to have attainable strict separation property if for every closed convex nonempty subset B of X with $A \cap B = \emptyset$ there exist $x^* \in X^*$, $a \in A$ and $b \in B$ such that*

$$\langle x^*, a\rangle = \inf_{x\in A}\langle x^*, x\rangle > \sup_{x\in B}\langle x^*, x\rangle = \langle x^*, b\rangle. \tag{3.10}$$

$(*)$ $\quad (3.9) \Longleftrightarrow \big[x^* \in N(B, b) \cap -N(A, a)\ \&\ \langle x^*, a - b\rangle \geq 0\big].$

Proposition 3.1. *Let A be a bounded closed convex set in a Banach space X. Then the following statements are equivalent:*

(i) A has the attainable separation property.

(ii) A has the attainable strict separation property.

(iii) A has the strict separation property.

(iv) A is weakly compact.

To consider the unbounded case , we adopt the following notion of an asymptotic hyperplane of A: *a hyperplane $\mathcal{P}(x^*, \alpha) := \{x \in X : \langle x^*, x\rangle = \alpha\}$ with $(x^*, \alpha) \in (X^* \setminus \{0\}) \times \mathbb{R}$ is called an asymptotic hyperplane of A if $\langle x^*, x\rangle \leq \alpha$ for all $x \in A$ (i.e., $\sigma_A(x^*) \leq \alpha$) and there exists a sequence $\{a_n\}$ in A such that*

$$\lim_{n\to\infty} \|a_n\| = \infty \quad \text{and} \quad \lim_{n\to\infty} d(a_n, \mathcal{P}(x^*, \alpha)) = 0.$$

Theorem 3.1. *Let X be a reflexive Banach space and A an unbounded closed convex subset of X. Then the following statements are equivalent:*
(i) A has the attainable strict separation property.
(ii) For every closed convex set B in X with $A \cap B = \emptyset$ there exist $a \in A$, $b \in B$ and $x^* \in N(B, b) \cap -N(A, a)$ such that $\|x^*\| = 1$ and

$$\langle x^*, a \rangle - \langle x^*, b \rangle = \|a - b\| = d(A, B).$$

(iii) A has no asymptotic hyperplane and $\mathrm{int}(A)$ is nonempty.
(iv) A is continuous and $\mathrm{int}(A)$ is nonempty.
(v) $A - B$ is closed for any closed convex set B disjoint with A.

Theorem 3.2. *Let X be a Banach space. Then the following statements are equivalent.*

(i) X is reflexive.

(ii) Every closed convex subset of X having no asymptotic hyperplane has the attainable separation property.

(iii) Every unbounded continuous closed convex subset of X having a nonempty interior has the attainable strict separation property.

(iv) There exist a closed subspace Y of X with $\operatorname{codim}(Y) = 1$ and an element e in $X \setminus Y$ such that

$$A(Y, e) := \{y + te : (y, t) \in Y \times \mathbb{R} \text{ and } \|y\|^2 \le t\} \tag{3.11}$$

has the attainable separation property.

(v) For any closed subspace Y of X with $\operatorname{codim}(Y) = 1$ and any element e in $X \setminus Y$, $A(Y, e)$ defined by (3.11) has the attainable strict separation property.

Proposition 3.2. *Let X be a finite-dimensional normed space and let A be a closed convex nonempty subset of X. Then the following statements are equivalent:*

(i) $\mathcal{S}(A, x^*)$ is a bounded nonempty set for each $x^* \in \mathrm{bar}(A) \setminus \{0\}$.

(ii) A has no asymptotic hyperplane.

(iii) A is continuous.

(iv) A has the attainable strict separation property.

(v) A has the attainable separation property.

(vi) A has the strict separation property.

(vii) $A - B$ is closed for every closed convex subset B of X.

(viii) $A - B$ is closed for every closed convex subset B of X with $\mathrm{int}(B) \neq \emptyset$ and $A \cap B = \emptyset$.

4 Well solvability of convex optimization problems

Theorem ETZ2 ([Ernst-Théra-Zalinescu, JFA, 2005]). *Let* X *be a reflexive Banach space and* $f : X \to \mathbb{R}$ *be a nonconstant continuous convex function such that* $f(x_0) = \min_{x\in X} f(x)$ *for some* $x_0 \in X$*. Then for any closed convex set* A *in* X *there exists* $a \in A$ *such that* $f(a) = \min_{x\in A} f(x)$ *if and only if* $f^{-1}(-\infty, \lambda]$ *is slice-continuous for all* $\lambda \geq \inf_{x\in X} f(x)$.

Remark. In Theorem ETZ2, the objective f is **a fixed continuous convex function**, while the constrained sets are **all closed convex sets** in the concerned space.

Next, we will consider, from a different angle than Theorem ETZ2, **a fixed closed convex set** A in a Banach space X such that **for every continuous (even lower semicontinuous) convex function** $f : X \to \mathbb{R}$ with $\inf_{x \in A} f(x) > -\infty$ the corresponding optimization problem

$$\mathcal{P}_A(f) \qquad \text{minimize } f(x) \quad \text{subject to } x \in A$$

is well solvable in the sense of various well-posedness.

Tychnov's well-posedness: *a proper lower semicontinuous extended-real function f on a normed space X is said to have the well-posedness property if every minimizing sequence $\{x_n\}$ of f (i.e. $\lim_{n\to\infty} f(x_n) = \inf_{x\in X} f(x)$) is convergent, while f is said to have the generalized well-posedness property if every minimizing sequence $\{x_n\}$ of f has a convergent subsequence.*

The well-posedness and generalized well-posedness have been recognized to be useful in optimization and studied extensively.

Definition 4.1 *Given a closed convex set A in a normed linear space X and a proper lower semicontinuous convex function $f : X \to \mathbb{R} \cup \{+\infty\}$ with $\inf\limits_{x\in A} f(x) > -\infty$, the corresponding constrained optimization problem $\mathcal{P}_A(f)$ is said to be*

(i) *well-posed-solvable if every minimizing sequence $\{x_n\}$ of $\mathcal{P}_A(f)$ (i.e., $\{x_n\} \subset A$ and $f(x_n) \to \inf\limits_{x\in A} f(x)$) is convergent;*

(ii) *$\mathcal{G}$-well-posed-solvable if every minimizing sequence of $\mathcal{P}_A(f)$ has a convergent subsequence;*

(iii) *$\mathcal{W}$-well-posed-solvable if every minimizing sequence $\{x_n\}$ of $\mathcal{P}_A(f)$ is weakly convergent;*

(iv) *$\mathcal{WG}$-well-posed-solvable if every minimizing sequence of $\mathcal{P}_A(f)$ has a weakly convergent subsequence;*

(v) *boundedly solvable if the solution set $\mathcal{S}(A,f) := \{a \in A : f(a) = \inf\limits_{x\in A} f(x)\}$ is bounded and nonempty.*

Proposition 4.1 *Let A be a closed convex set in a normed linear space X and let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous continuous convex function. Then the following statements hold:*
(i) $\mathcal{P}_A(f)$ is $\mathcal{G}$-well-posed-solvable if and only if the solution set $\mathcal{S}(A, f)$ is a compact nonempty set and $\mathrm{d}(x_n, \mathcal{S}(A, f)) \to 0$ for every minimizing sequence $\{x_n\}$ of $\mathcal{P}_A(f)$.
(ii) $\mathcal{P}_A(f)$ is $\mathcal{WG}$-well-posed-solvable if and only if $\mathcal{S}(A, f)$ is a weak-compact nonempty set and every minimizing sequence $\{x_n\}$ of $\mathcal{P}_A(f)$ converges to $\mathcal{S}(A, f)$ with respect to the weak topology, that is, for any weak neighborhood U of 0 there exists N such that $x_n \in \mathcal{S}(A, f) + U$ for all $n \geq N$.

The main aims of this talk are to study the following two topics:

(T1) *Characterize a given closed convex set A in a Banach space X such that for every convex continuous function $f : X \to \mathbb{R}$ with $\inf_{x \in A} f(x) > -\infty$ the corresponding optimization problem $\mathcal{P}_A(f)$ is well-posed solvable, $\mathcal{G}$-well-posed solvable or $\mathcal{WG}$-well-posed solvable.*

(T2) *Find some conditions on a given real-valued continuous convex function f on a Banach space X such that for every closed convex subset A of X the corresponding optimization problem $\mathcal{P}_A(f)$ is solvable or well-posed solvable.*

4.1. Slice property, continuity and differentiability

Let A be a closed convex set in a normed space X. Recall that the support functional and the bar cone of A are respectively defined by

$$\sigma_A(x^*) := \sup_{x\in A}\langle x^*, x\rangle \quad \forall x^* \in X^*$$

and

$$\mathrm{bar}(A) := \mathrm{dom}(\sigma_A) = \{x^* \in X^* :\ \sigma_A(x^*) < +\infty\}.$$

For $x^* \in \mathrm{bar}(A)$ and $\varepsilon > 0$, the corresponding support set and slice of A are defined as

$$\mathcal{S}(A, x^*) := \{x \in A :\ \langle x^*, x\rangle = \sigma_A(x^*)\}$$

and

$$\mathcal{S}(A, x^*, \varepsilon) := \{x \in A :\ \langle x^*, x\rangle \geq \sigma_A(x^*) - \varepsilon\}.$$

It is clear that $\mathcal{S}(A, x^*) = \bigcap_{\varepsilon>0} \mathcal{S}(A, x^*, \varepsilon)$.

Definition 4.2 *A closed convex set A in a normed space X is said to have*
(i) bounded slice property if for each $x^* \in \mathrm{bar}(A) \setminus \{0\}$ there exists $\varepsilon > 0$ such that $\mathcal{S}(A, x^*, \varepsilon)$ is bounded, and
(ii) strong slice property if $\lim\limits_{\varepsilon \to 0^+} \mathrm{diam}(\mathcal{S}(A, x^*, \varepsilon)) = 0$ for all $x^* \in \mathrm{bar}(A) \setminus \{0\}$, where $\mathrm{diam}(\mathcal{S}(A, x^*, \varepsilon)) := \sup\{\|x_1 - x_2\| : \ x_1, x_2 \in \mathcal{S}(A, x^*, \varepsilon)\}$.

Lemma 4.1 *Let A be a closed convex set in a normed space X. The following statements hold:*
(i) $\mathcal{S}(A, x^*, \varepsilon) \subset \partial\sigma_A(B(x^*, \sqrt{\varepsilon})) + \sqrt{\varepsilon} B_{X^{**}} \quad \forall (x^*, \varepsilon) \in \mathrm{bar}(A) \times (0, \ +\infty)$, where $B_{X^{**}}$ denotes the unit ball of the bidual space X^{**}.
(ii) For any $x^* \in \mathrm{bar}(A) \setminus \{0\}$ there exist $\varepsilon_0, L_0 \in (0, \ +\infty)$ such that

$$\partial\sigma_A(B(x^*, \varepsilon)) \subset \overline{\mathcal{S}(A, x^*, L_0\varepsilon)}^{w^*} \quad \forall \varepsilon \in (0, \ \varepsilon_0).$$

Consequently $\lim\limits_{\varepsilon \to 0^+} \mathrm{diam}(\partial\sigma_A(B(x^*, \varepsilon)) = \lim\limits_{\varepsilon \to 0^+} \mathrm{diam}(\mathcal{S}(A, x^*, \varepsilon))$ for all $x^* \in \mathrm{bar}(A) \setminus \{0\}$.

Proposition 4.2. *Let A be a closed convex set in a normed space X and let $x_0^* \in \mathrm{bar}(A) \setminus \{0\}$. Then the following statements are equivalent.*

(i) $\mathcal{S}(A, x_0^*, \varepsilon)$ is bounded for all $\varepsilon \in (0, +\infty)$.

(ii) There exists $\varepsilon_0 > 0$ such that $\mathcal{S}(A, x_0^*, \varepsilon_0)$ is bounded.

(iii) $x_0^* \in \mathrm{int}(\mathrm{bar}(A))$.

(iv) σ_A is continuous at x_0^*.

(v) There exist $\varepsilon_0, \delta_0 \in (0, +\infty)$ such that

$$\sup\left\{\|x\| : x \in \bigcup_{x^* \in B(x_0^*, \delta_0)} \mathcal{S}(A, x^*)\right\} < +\infty.$$

Proposition 4.3. *Let A be a closed convex set in a finite-dimensional normed space X and let $x_0^* \in \operatorname{bar}(A) \setminus \{0\}$ be such that the support set $\mathcal{S}(A, x_0^*)$ is bounded and nonempty. Then the slice $\mathcal{S}(A, x_0^*, \varepsilon)$ is bounded for all $\varepsilon > 0$, and*

$$\lim_{\varepsilon \to 0^+} \sup_{x \in \mathcal{S}(A, x_0^*, \varepsilon)} d(x, \mathcal{S}(A, x_0^*)) = 0.$$

Consequently, $\mathcal{S}(A, x_0^*)$ is a singleton if and only if $\lim_{\varepsilon \to 0^+} \operatorname{diam}(\mathcal{S}(A, x_0^*, \varepsilon)) = 0$.

Theorem 4.1 *Let A be a closed convex set in a normed space X. Then the following statements are equivalent:*

(i) A is continuous.

(ii) $\operatorname{bar}(A) \setminus \{0\}$ is open.

(iii) A has the bounded slice property.

Definition 4.3 *A closed convex set A in a normed space X is said to be differentiable if its support functional σ_A is differentiable at each point of $\mathrm{dom}(\sigma_A)\setminus\{0\}$.*

Every closed ball in a Hilbert space is differentiable.

Example 4.1. *Let X be a Hilbert space. Then, for any $e \in X \setminus \{0\}$ and $p \in (1, +\infty)$, $A(e,p) := \{x + te : x \in e^{\perp} \ \&\ \|x\|^p \le t\}$ is differentiable, where $e^{\perp} = \{x \in X : \langle x, e\rangle = 0\}$.*

Proposition 4.4 *Let A be a closed convex set in a normed space X. Then A has the strong slice property if and only if A is differentiable.*

Recall that A is said to be a Chebychev set (or to have the Chebychev property) if for each $x \in X$ there exists $a \in A$ such that $d(x, A) = \|x - a\|$. To characterize further the strong slice property, we adopt the following notion: A *is said to have the S-Chebychev property if for every closed convex set* B *with* $d(A, B) > 0$ *there exists a unique* $a \in A$ *such that* $d(a, B) = d(A, B)$ *and* $\lim_{n\to\infty} \|a_n - a\| = 0$ *for any sequence* $\{a_n\} \subset A$ *with* $\lim_{n\to\infty} d(a_n, B) = d(A, B)$.

Proposition 4.5 *Given a closed convex set* A *in a Banach space* X*, the following statements hold:*

(i) A *is differentiable if and only if* A *has the S-Chebychev property.*

(ii) If, in addition, $\text{int}(A) \neq \emptyset$*, then* A *is differentiable if and only if for every closed convex set* B *disjoint with* A *there exists a unique* $a \in A$ *such that* $d(a, B) = d(A, B)$ *and* $\lim_{n\to\infty} \|a_n - a\| = 0$ *for any sequence* $\{a_n\} \subset A$.

Proposition 4.6 *Let X be a Banach space and Y be a closed subspace of X such that* $\operatorname{codim}(Y) = 1$. *For $e \in X \setminus Y$ and $p \in (1, +\infty)$, let*

$$A_p(Y, e) := \{y + te : \ y \in Y \text{ and } \|y\|^p \leq t\}. \tag{4.12}$$

Then the following statements hold:

(i) $A_p(Y, e)$ *has the bounded slice property and* $\operatorname{int}(A_p(Y, e)) \neq \emptyset$.

(ii) *If, in addition, X is reflexive and locally uniformly convex, $A_p(Y, e)$ has the strong slice property.*

4.2. Main Results

For a closed convex set A in a normed space X, we adopt the following notation

$$\mathfrak{L}(X|A) := \{u^* \in X^* \setminus \{0\} : \inf_{x \in A} \langle u^*, x \rangle > -\infty\}.$$

Let $\mathfrak{C}(X|A)$ denote the family of all continuous convex functions $f : X \to \mathbb{R}$ satisfying $\inf_{x \in A} f(x) > \inf_{x \in X} f(x)$.

$$\mathfrak{L}(X|A) \subset \mathfrak{C}(X|A).$$

Lemma 4.2. *Let A be a closed convex set in a normed space X. Then, for each $f \in \mathfrak{C}(X|A)$, there exists $u_f^* \in \mathfrak{L}(X|A)$ such that every minimizing sequence of the convex optimization problem $\mathcal{P}_A(f)$ is a minimizing sequence of the linear optimization problem $\mathcal{P}_A(u_f^*)$.*

Theorem 4.2. *Let A be a closed convex set in a Banach space X. Then the following statements are equivalent:*

(i) A is differentiable.

(ii) For any $u^* \in \mathfrak{L}(X|A)$, the corresponding linear optimization problem $\mathcal{P}_A(u^*)$ is well-posed-solvable.

(iii) For any $f \in \mathfrak{C}(X|A)$, the corresponding convex optimization problem $\mathcal{P}_A(f)$ is well-posed-solvable.

Theorem 4.3 *Let A be a closed convex set in a finite dimensional normed space X. Then the following statements are equivalent:*

(i) A is differentiable.

(ii) For any $u^* \in \mathfrak{L}(X|A)$, the corresponding linear optimization problem $\mathcal{P}_A(u^*)$ has a unique solution.

(iii) For every proper lower semicontinuous convex function $f : X \to \mathbb{R} \cup \{+\infty\}$ with $\inf_{x\in A} f(x) > \inf_{x\in X} f(x)$, the corresponding convex optimization problem $\mathcal{P}_A(f)$ is well-posed-solvable.

Theorem 4.4 *Let A be a closed convex set in a reflexive Banach space X. Then the following statements are equivalent:*

(i) A is continuous.

(ii) For any $u^* \in \mathfrak{L}(X|A)$, the corresponding linear optimization problem $\mathcal{P}_A(u^*)$ is $\mathcal{WG}$-well-posed-solvable.

(iii) For any $f \in \mathfrak{C}(X|A)$, the corresponding convex optimization problem $\mathcal{P}_A(f)$ is $\mathcal{WG}$-well-posed-solvable.

James Theorem ([Ann. Math. 1957] and [Trans. Amer. Math. Soc. 1964]). *Let X be a Banach space X. Then X is reflexive if and only if the closed unit ball B_X is weakly compact if and only if for any bounded closed convex set $A \subset X$ and any $x^* \in X^*$, the linear optimization problem $\mathcal{P}_A(x^*)$ is solvable.*

Theorem 4.5. *Let X be a reflexive Banach space and let A be an unbounded closed convex subset of X such that $\mathrm{int}(A) \neq \emptyset$. Then A is continuous if and only if for every proper lower semicontinuous convex function $f : X \to \mathbb{R} \cup \{+\infty\}$ with $\inf_{x \in A} f(x) > \inf_{x \in X} f(x)$, the corresponding convex optimization problem $\mathcal{P}_A(f)$ is $\mathcal{WG}$-well-posed-solvable.*

Theorem 4.6. *Let A be a closed convex subset of a finite dimensional normed space X. Then the following statements are equivalent:*

(i) A is continuous.

(ii) For each $u^* \in \mathfrak{L}(X|A)$, the corresponding linear optimization problem $\mathcal{P}_A(u^*)$ is boundedly solvable.

(iii) For every proper lower semicontinuous convex function $f : X \to \mathbb{R} \cup \{+\infty\}$ with $\inf\limits_{x \in A} f(x) > \inf\limits_{x \in X} f(x)$, the corresponding convex optimization problem $\mathcal{P}_A(f)$ is $\mathcal{G}$-well-posed-solvable.

4.3. Differtiabilty and continuity of conjugate functions

Recall the conjugate function f^* of f defined by

$$f^*(x^*) := \sup_{x \in X}(\langle x^*, x\rangle - f(x)) \quad \forall x^* \in X^*.$$

It is well known that the conjugate function f^* is always lower semicontinuous with respect to the weak* topology on X^* and useful in convex optimization.

Theorem 4.7. *Let X be a Banach space and $f : X \to \mathbb{R}$ be a continuous convex function such that f^* is Fréchet differentiable $\mathrm{dom}(f^*)$. Then, for every closed convex subset A of X with $-\infty < \inf_{x \in A} f(x)$, the corresponding convex optimization problem $\mathcal{P}_A(f)$ is well-posed solvable.*

Theorem 4.8. *Let X be a reflexive Banach space and $f : X \to \mathbb{R}$ be a continuous convex function such that f^* is is continuous on $\mathrm{dom}(f^*)$. Then, for every closed convex subset A of X with $\inf_{x \in A} f(x) > -\infty$, the corresponding optimization problem $\mathcal{P}_A(f)$ is $\mathcal{WG}$-well-posed solvable.*

Proposition 4.7. *Let X be a normed space and $f : X \to \mathbb{R}$ be a continuous convex function. Then $\mathrm{epi}(f)$ is differentiable if and only if f^* is Fréchet differentiable on $\mathrm{dom}(f^*)$.*

Proposition 4.8. *Let X be a normed space and $f : X \to \mathbb{R}$ be a continuous convex function. Then the following statements are equivalent:*
(i) $\mathrm{epi}(f)$ *is continuous.*
(ii) f^* *is continuous on* $\mathrm{dom}(f^*)$.
(iii) $\mathrm{dom}(f^*)$ *is open.*

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