OPTIMAL CONTROL OF SWEEPING PROCESSES WITH APPLICATIONS TO ROBOTICS AND TRAFFIC EQUILIBRIA

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CONTROLLED SWEEPING PROCESS

Denote by (P) the following optimal control problem

$$\text{minimize} \quad J[x,u] := \varphi\big(x(T)\big)$$

over pairs $(x(\cdot),u(\cdot))$ of measurable controls $u(t)$ and absolutely continuous trajectories $x(t)$ on the time interval $[0,T]$ satisfying the perturbed controlled sweeping/Moreau differential inclusion

$$\dot{x}(t) \in -N\big(x(t);C\big) + g\big(x(t),u(t)\big) \text{ a.e. } t \in [0,T], x(0) := x_0 \in C \subset I\!R^n$$

subject to the pointwise constraints on control functions

$$u(t) \in U \subset I\!R^d \;\text{ a.e. }\; t \in [0,T]$$

The sweeping set C is a convex polyhedron given by

$$C := \bigcap_{j=1}^{s} C^j \;\text{ with }\; C^j := \Big\{x \in I\!R^n \Big| \langle x_*^j, x\rangle \le c_j\Big\}$$

and the normal cone to it in any $x \in I\!R^n$ is understood in the classical sense of convex analysis

$$N\Big(x;C\Big) := \Big\{v \in I\!R^n \Big| \ \langle v, y - x\rangle \le 0,\ y \in C\Big\} \text{ if } x \in C$$
$$N\Big(x;C\Big) := \emptyset \ \text{ otherwise}$$

The latter yields the pointwise state constraints

$$\langle x_*^j, x(t)\rangle \le c_j \ \text{ for all } \ t \in [0,T], \quad j = 1,\ldots,s$$

Problem (P) belongs to the most difficult one in control theory being governed by discontinuous differential inclusions with the simultaneous presence of hard/pointwise constraints on both state and control functions

Some partial results on necessary optimality conditions for (P) were obtained during very recent years in the cases where either $U = I\!R^d$ with absolutely continuous controls (Cao-BM 2016-19), or when C is strictly convex and smooth of higher order (Arround-Colombo 2018, de Pinho et al. 2019)

FEASIBLE AND LOCALLY OPTIMAL SOLUTIONS

By a feasible solution to (P) we understand a pair $(u(\cdot), x(\cdot))$ such that $u(\cdot)$ is measurable and that $x(\cdot) \in W^{1,2}([0,T], I\!R^n)$ subject to the above constraints. The set of feasible solutions is nonempty under mild assumptions.

DEFINITION A feasible pair $(\bar{x}(\cdot), \bar{u}(\cdot))$ for (P) is a $W^{1,2} \times L^2$-local minimizer for this problem if there is $\varepsilon > 0$ such that $J[\bar{x}, \bar{u}] \leq J[x, u]$ for all the feasible pairs $(x(\cdot), u(\cdot))$ satisfying

$$\int_0^T \Big(\|\dot{x}(t) - \dot{\bar{x}}(t)\|^2 + \|u(t) - \bar{u}(t)\|^2 \Big) dt < \varepsilon$$

It is clear that this notion of local minimizers for (P) includes, in the framework of sweeping control problems, strong $\mathcal{C} \times L^2$-local minimizers and occupies an intermediate position between the conventional notions of strong and weak minima in the calculus of variations and optimal control

STANDING ASSUMPTIONS

The listed assumptions are essentially simplified in comparison be those in [Colombo-BM-Nguyen 2020]

(H1) The control set U is compact and convex in $I\!R^d$, and the image set $g(x,U)$ is convex in $I\!R^n$

(H2) The cost function $\varphi\colon I\!R^n \to I\!R$ is $\mathcal{C}^1$-smooth around $\bar{x}(T)$

(H3) The perturbation mapping $g\colon I\!R^n \times I\!R^d \to I\!R^n$ is $\mathcal{C}^1$-smooth around $(\bar{x}(\cdot),\bar{u}(\cdot))$ and satisfies the sublinear growth condition

$$\|g(x,u)\| \le \beta\Big(1+\|x\|\Big) \quad \text{for all} \quad u \in U \quad \text{with some} \quad \beta > 0$$

(H4) The vertices x_*^j of the polyhedron satisfy the linear independence constraint qualification

$$\Big[\sum_{j\in I(\bar{x})}\alpha_j x_*^j = 0,\ \alpha_j\in I\!\!R\Big] \Longrightarrow \Big[\alpha_j = 0 \ \text{ for all } \ j\in I(\bar{x})\Big\}$$

along the trajectory $\bar{x}=\bar{x}(t)$ as $t\in[0,T]$, where $I(\bar{x}) := \{j\in \{1,\ldots,s\} \mid \langle x_*^j,\bar{x}\rangle = c_j\}$

DISCRETE APPROXIMATIONS OF FEASIBLE SOLUTIONS

Given any $m \in I\!N := \{1, 2, \ldots\}$, consider the discrete mesh

$\Delta_m := \Big\{0 = t_{0m} < t_{1m} < \ldots < t_{2^m m} = T\Big\}$ with $h_m := t_{(k+1)m} - t_{km}$ on $[0, T]$ and the sequence of discrete-time inclusions approximating the controlled sweeping process

$x_{(k+1)m} \in x_{km} + h_m\Big(g(x_{km}, u_{km}) - N(x_{km}; C)\Big)$ as $k = 0, \ldots, 2^m - 1$

over discrete pairs $(x_m, u_m) = (x_{1m}, \ldots, x_{2^m m}, u_{0m}, u_{1m}, \ldots, u_{(2^m-1)m})$ with $x_{0m} = x_0 \in C$ and the control constraints

$$u_m = \Big(u_{0m}, u_{1m}, \ldots, u_{(2^m-1)m}\Big) \in U$$

Denote by $I_{km} := [t_{(k-1)m}, t_{km})$ for $k = 1, \ldots, 2^m$ the corresponding subintervals of $[0, T]$

STRONG CONVERGENCE FOR FEASIBLE SOLUTIONS

Let $(x(\cdot), u(\cdot))$ be a feasible solution to (P)

THEOREM Assume that $\bar{u}(\cdot)$ is of bounded variation (BV) with a right continuous representative on $[0, T]$. Then there exist sequences of unit vectors sequences $z_m^{jk} \to x_*^j$, vectors $c_m^{jk} \to c_j$ as $m \to \infty$, and state-control pairs $(\bar{x}_m(t), \bar{u}_m(t))$, $0 \leq t \leq T$, for which we have:

(a) The sequence of controls $\bar{u}_m : [0, T] \to U$, which are constant on each interval I_{km}, converges to $\bar{u}(\cdot)$ strongly in $L^2([0, T]; I\!R^d)$

(b) The sequence of continuous state mappings $\bar{x}_m : [0, T] \to I\!R^n$, which are affine on each interval I_{km}, converges strongly in $W^{1,2}([0, T]; I\!R^n)$ to $\bar{x}(\cdot)$ and satisfy the inclusions

$\bar{x}_m(t_{km}) = \bar{x}(t_{km}) \in C_{km}$ for each $k = 1, \ldots, 2^m$ with $\bar{x}_m(0) = x_0$

where the perturbed polyhedra C_{km} are given by

$$C_{km} := \bigcap_{j=1}^{s} \Big\{ x \in I\!\!R^n \;\Big|\; \langle z_m^{jk}, x \rangle \le c_m^{jk} \Big\}, \quad k = 1, \ldots, 2^m, \; C_{0m} := C$$

(c) For all $t \in (t_{(k-1)m}, t_{km})$ and $k = 1, \ldots, 2^m$ we have

$$\dot{\bar{x}}_m(t) \in -N\Big(\bar{x}_m(t_{km}); C_{km}\Big) + g\Big(\bar{x}_m(t_{km}), \bar{u}_m(t)\Big)$$

DISCRETE APPROXIMATIONS OF OPTIMAL SOLUTIONS

THEOREM Let $(x(\cdot),u(\cdot))$ be a $W^{1,2}\times L^2$-local minimizer to (P) in the framework of the previous theorem. Then for each $m\in I\!N$ the pair $(\bar{x}_m(\cdot),\bar{u}_m(\cdot))$ can be chosen so that its restriction on the discrete mesh Δ_m is an optimal solution to the discrete sweeping control problem (P_m) of minimizing the cost functional

$$J_m[x_m,u_m]:\ =\varphi\Big(x_m(T)\Big)+\frac{1}{2}\sum_{k=0}^{2^m-1}\int_{t_{km}}^{t_{(k+1)m}}\left(\Big\|\frac{x_{(k+1)m}-x_{km}}{h_m}-\dot{\bar{x}}(t)\Big\|^2\right.$$
$$\left.+\Big\|u_{km}-\bar{u}(t)\Big\|^2\right)dt$$

over all pair (x_m,u_m) satisfying the above constraints and the $W^{1,2}\times L^2$-localization constraint

$$\sum_{k=0}^{2^m-1}\int_{t_{km}}^{t_{(k+1)m}}\left(\Big\|\frac{x_{(k+1)m}-x_{km}}{h_m}-\dot{\bar{x}}(t)\Big\|^2+\Big\|u_{km}-\bar{u}(t)\Big\|^2\right)dt\le\frac{\varepsilon}{2}$$

NECESSARY CONDITIONS FOR SWEEPING PROCESSES

THEOREM Let $(\bar{x}(\cdot), \bar{u}(\cdot))$ be a $W^{1,2} \times L^2$-local minimizer for (P). Then there exist a multiplier $\lambda \geq 0$, a measure $\gamma = (\gamma^1, \ldots, \gamma^n) \in C^*([0,T]; I\!R^n)$, adjoint arcs $p(\cdot) \in W^{1,2}([0,T]; I\!R^n)$, $q(\cdot) \in BV([0,T]; I\!R^n)$ such that $\lambda + \|q(0\| + \|p(T)\| > 0$ and the following conditions are satisfied

- Primal velocity representation

$$-\dot{\bar{x}}(t) = \sum_{j=1}^{s} \eta^j(t) x_*^j - g\Big(\bar{x}(t), \bar{u}(t)\Big) \text{ for a.e. } t \in [0,T]$$

where $\eta^j(\cdot) \in L^2([0,T]; I\!R_+)$ is uniquely determined by and well defined at $t = T$

- Adjoint system

$$\dot{p}(t) = -\nabla_x g\Big(\bar{x}(t), \bar{u}(t)\Big)^* q(t) \text{ for a.e. } t \in [0,T]$$

where the dual arcs $q(\cdot)$ and $p(\cdot)$ are precisely connected by

$$q(t) = p(t) - \int_{(t,T]} d\gamma(\tau)$$

which holds for all $t \in [0, T]$ except at most a countable subset

• Maximization condition

$$\Big\langle \psi(t), \bar{u}(t) \Big\rangle = \max \Big\{ \Big\langle \psi(t), u \Big\rangle \Big| \, u \in U \Big\} \ \text{ with } \ \psi(t) := \nabla_u g\Big(\bar{x}(t), \bar{u}(t)\Big)^* q(t)$$

• Complementarity conditions

$$\Big\langle x_*^j, \bar{x}(t) \Big\rangle < c_j \Longrightarrow \eta^j(t) = 0 \ \text{ and } \ \eta^j(t) > 0 \Longrightarrow \Big\langle x_*^j, q(t) \Big\rangle = c_j$$

for a.e. $t \in [0, T]$ including $t = T$ and for all $j = 1, \ldots, s$

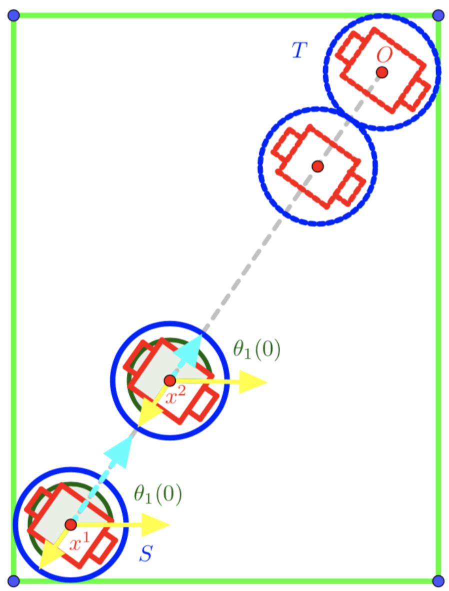
• Right endpoint transversality conditions

$$-p(T) = \lambda \nabla \varphi\Big(\bar{x}(T)\Big) + \sum_{j \in I(\bar{x}(T))} \eta^j(T) x_*^j, \quad \sum_{j \in I(\bar{x}(T))} \eta^j(T) x_*^j \in N\Big(\bar{x}(T); C\Big)$$

• Measure nonatomicity condition: If $t \in [0, T)$ and $\langle x_*^j, \bar{x}(t) \rangle < c_j$ for all $j = 1, \ldots, s$, then there is a neighborhood V_t of t in $[0, T]$ such that $\gamma(V) = 0$ for all the Borel subsets V of V_t

CONTROLLED MOBILE ROBOT MODEL WITH OBSTACLES

This model concerns n mobile robots ($n \geq 2$) identified with safety disks in the plane of the same radius R. A simulation/uncontrolled version of it was suggested by Hedjar and Bounkhel (2014). The goal of each robot is to reach the target by the shortest path during a fixed time interval $[0, T]$ while avoiding the other $n-1$ robots that are treated by as obstacles



SWEEPING CONTROL DESCRIPTION OF ROBOT MODEL

$$\text{minimize } J[x,u] := \frac{1}{2}\Big\|x(T)\Big\|^2$$

subject to the constraints

$$\begin{cases} -\dot{x}(t) \in N\Big(x(t);C\Big) - g\Big(x(t),u(t)\Big) \\ x(0) = x_0 \in C,\ u(t) \in U \ \text{ a.e. on } [0,T] \end{cases}$$

where $x = (x^1,\ldots,x^n) \in I\!R^{2n}$, $u = (u^1,\ldots,u^n) \in I\!R^n$,

$$g\Big(x(t)\Big) := -\Big(s_1 \cos\theta_1, s_1 \sin\theta_1, \ldots, s_n \cos\theta_n, s_n \sin\theta_n\Big) \in I\!R^{2n}$$

where s_i denotes the speed of robot i under the crucial noncollision condition in contact

$$\|x^i - x^j\| \geq R \ \text{ for all } \ i,j \in \{1,\ldots,n\}$$

The sweeping set C is defined by

$$C := \left\{x \in I\!R^{2n} \middle| \; \langle x_*^j, x\rangle \le c_j, \; j = 1, \ldots, n-1\right\}$$

with $c_j := -2R$ and with the $n-1$ vertices of the polyhedron

$$x_*^j := e_{j1} + e_{j2} - e_{(j+1)1} - e_{(j+1)2} \in I\!R^{2n}, \quad j = 1, \ldots, n-1$$

where e_{ji} are the vectors in the form

$$e := \left(e_{11}, e_{12}, e_{21}, e_{22}, \ldots, e_{n1}, e_{n2}\right) \in I\!R^{2n}$$

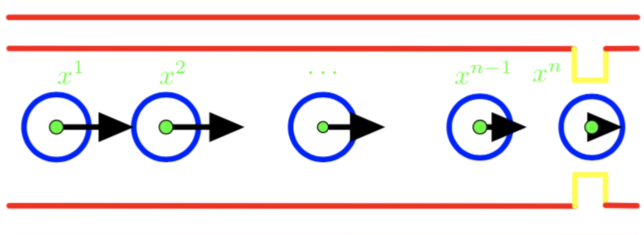
with 1 at only one position of e_{ji} and 0 at all the other positions

The obtained necessary optimality conditions allow us to derive verifiable relationships for optimal controls and trajectories in generality and then completely solve the model in the case of one obstacle

CONTROLLED MODEL OF PEDESTRIAN TRAFFIC FLOWS

Now we formulate a continuous-time, deterministic, and optimal control version of the pedestrian traffic flow model through a doorway for which a stochastic, discrete-time, and simulation (uncontrolled) counterpart was originated by Lovas (1994). Here we formalize the dynamics via a perturbed sweeping process with constrained controls in perturbations that should be determined to ensure the desired performance

In the model we have n pedestrians $x^i \in I\!R,\ i = 1, \ldots, n$ as $n \geq 2$ that are identified with rigid disks of the same radius R going through a doorway



DESCRIPTION VIA A CONTROLLED SWEEPING PROCESS

Sweeping dynamics

$$\dot{x}(t) \in -N(x;Q_0) + S(x) \text{ for a.e. } t \in [0,T], \quad x(0) = x_0$$

where Q_0 is the set of admissible configurations given via nonoverlapping conditions by

$$Q_0 := \Big\{ x = \big(x^1, \ldots, x^n\big) \in \mathbb{R}^n \Big| \, x^{i+1} - x^i \geq 2R \text{ for all } i, j \in \{1, \ldots, n\} \Big\}$$

and where $S(x)$ is the spontaneous velocity of the pedestrians at $x \in Q_0$

$$S(x) := \big(S_0(x^1), \ldots, S_0(x^n)\big) \text{ with } x; S_0(x) = s_0 \nabla D(x), \quad x \in Q_0$$

with $D(x)$ standing for the distance from the position $x = (x^1, \ldots, x^n) \in Q_0$ to the doorway and with $s_0 = \|S_0(x)\|$. The additive control term is described by

$$g\big(x(t), u(t)\big) := \big(s_1 u^1(t), \ldots, s_n u^n(t)\big), \quad t \in [0,T]$$

where s_i denotes the speed of the pedestrian $i \in \{1, \ldots, n\}$. The main difference from the crowd motion model is the presence of constraints on controls in the form

$$u(t) \in U \quad \text{a.e. on} \quad [0, T] \tag{1}$$

defined via a specified convex and compact set $U \subset I\!R^n$. The The cost functional is

$$\text{minimize} \quad J[x, u] := \frac{1}{2} \|x(T)\|^2$$

meaning the minimization of the distance from all the pedestrians to the doorway at the origin

The obtained necessary optimality conditions involving the new Maximum Principle provide verifiable relationships in the general model which allow us to fully calculate optimal controls for models with two and three participants

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