THE AX-LINDEMANN-WEIERSTRASS THEOREM FOR QUOTIENTS OF BOUNDED SYMMETRIC DOMAINS BY ARBITRARY COCOMPACT LATTICES

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Abstract: Let $\Omega \Subset \mathbb{C}^N$ be a bounded symmetric domain in its Harish-Chandra realization and $\Gamma \subset \mathrm{Aut}(\Omega)$ be a torsion-free cocompact lattice. Define $X_\Gamma := \Omega/\Gamma$, which carries naturally the structure of a quasi-projective manifold, and write $\pi : \Omega \to X_\Gamma$ for the uniformization map. Let $Z \subset \Omega$ be an irreducible algebraic subset in the sense that Z is an irreducible component of $Z' \cap \Omega$ for some affine-algebraic subset $Z' \subset \mathbb{C}^N$. When $\Gamma \subset \mathrm{Aut}(\Omega)$ is an *arithmetic* lattice, the Ax-Lindemann-Weierstrass theorem of Klingler-Ullmo-Yafaev (2016) says that the Zariski closure $Y := \overline{\pi(Z)}^{\mathrm{Zar}}$ of $\pi(Z) \subset X_\Gamma$ in X_Γ is necessarily a totally geodesic subset. Mok-Pila-Tsimerman (2019) proved the Ax-Schanuel theorem for arithmetic lattices Γ, which is a theorem on the transcendence degrees of function fields obtained by restricting Harish-Chandra coordinates and Γ-equivariant modular functions on Ω to germs of complex-analytic subvarieties $(V; x)$ on Ω, a result superseding Ax-Lindemann-Weierstrass. The existing proofs of both Ax-Lindemann-Weierstrass and Ax-Schanuel for arithmetic lattices Γ require the use of the counting theorem of Pila-Wilkie in o-minimal geometry, a theory belonging to model theory in mathematical logic. The counting theorem is not applicable in the general case of nonarithmetic lattices, e.g., for most lattices in $\mathrm{Aut}(\Omega)$ when Ω is reducible and it has 1-dimensional irreducible factors.

It is desirable to remove the *arithmeticity assumption* for lattices $\Gamma \subset \mathrm{Aut}(\Omega)$ in Ax-type results. While the general case of Ax-Schanuel remains difficult, we are now able to prove the Ax-Lindemann-Weierstrass theorem for *arbitrary* cocompact lattices Γ. A special case of a *uniformization theorem* of Chan-Mok (2022) proves total geodesy of $Y \subset X_\Gamma$ when $\pi(Z) \subset X_\Gamma$ is Zariski closed so that $Y = \pi(Z)$. For Ax-Lindeman-Weierstrass in general we extend the foliation-theoretic approach of Mok (2019) which established the theorem for all lattices in the rank-1 case. By applying the rescaling method to a certain subvariety $Z' \subset \Omega$ derived from some foliation we show that Z' decomposes into a union of holomorphic isometric copies S_t of complex unit balls into Ω, noting

that these are subsets of fibers of some canonical map associated to an inverse partial Cayley transform. The latter allows us to generate a one-parameter group T of translations on Z', which serves as the starting point for proving a critical intermediate result asserting the normality of a maximal algebraic subgroup $H \subset \mathrm{Aut}(\Omega)$ leaving Z' invariant. Our proof also strengthens the result of Chan-Mok by requiring only that Z extends analytically beyond $\overline{\Omega}$ in place of requiring algebraicity of $Z \subset \Omega$.